

Review of the PhD thesis of Aidyn Kassymov  
*“Basic functional and geometric inequalities  
for the fractional order operators on  
homogenous Lie groups”*

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**1 The relevance of the research topic and its relationship with general scientific and national programs**

In 1974, Folland G. B., Stein E. M. noticed that the Heisenberg group and the functional spaces on it are closely related to boundary value problems for pseudoconvex domains in  $C^n$ . Further interest in the Heisenberg group was caused by the peculiar anisotropy that functional spaces on the Heisenberg group possess. The existing parallel between an odd-dimensional Euclidean space and a Heisenberg group makes it possible to compare "Heisenberg" and Euclidean anisotropy. On the one hand, the Heisenberg group is an odd-dimensional manifold and its Atlas consists of just one map. On the other hand, it lacks a Riemannian and affine structure. Nevertheless, Taylor M. E. was able to construct a fairly advanced noncommutative harmonic analysis on the Heisenberg group. The tangent space of a Heisenberg group at a unit point represents an algebra and its basis consists of first-order differential operators. An analogue of the Euclidean Laplacian is introduced as standard on this basis. Sublaplacian is a differential operator of second order. In the work of Folland G. B., Stein E. M., the fundamental solution of the suplaplacian is defined. Then, using convolution, an analogue of the Newtonian potential is introduced. Analogues of Riesz and Bessel potentials on Heisenberg groups are also defined, since the suplaplacian is a positive-defined essentially self-adjoint operator. Since the Haar measure on the Heisenberg group coincides with the Lebesgue measure, the introduced function spaces have similar properties that are inherent in standard function spaces. The next natural tendency is to transfer results on Heisenberg groups to objects defined on homogeneous as well as stratified groups. Therefore, the relevance of the research topic is not in doubt.

## **2 Scientific results and their validity**

The PhD dissertation consists of Introduction, Preliminaries, three chapters, Appendix, Conclusion and References. In Chapter 3 of the thesis, the theory of fractional functional and geometric inequalities on homogeneous Lie groups is developed. The PhD candidate obtained Hardy, Sobolev, Gagliardo-Nirenberg, Caffarelli-Kohn-Nirenberg inequalities on the homogeneous Lie groups. In the specified inequalities, the ranges of changes included in the parameter inequalities are much expanded.

Chapter 4 examines the reversibility of functional inequalities proved in Chapter 3. The results obtained by the author in this direction are new even in the case of Euclidean space.

It is remarkable that the author has demonstrated the various applications of the inequalities obtained in chapters 3 and 4. Chapter 5 shows the application of the Hardy and Sobolev inequalities from Chapter 3 to estimate the first eigenvalue of a fractional-order suplaplacian. A generalization of one Lyapunov inequality is also used. It is important that the results here concern nonlinear equations.

## **3 The degree of validity and reliability of each scientific result, proof and conclusions of the applicant, formulated in the dissertation**

The results and conclusions given in the dissertation are fully justified and given with complete and accurate proof. Well-known methods of functional analysis, function theory, and non-commutative analysis are used. The results of chapters 3, 4, and 5 are completely new or have elements of novelty and are published in rated international mathematical journals. There is a sufficient number of publications and their quality is not objectionable.

## **4 The degree of novelty of each scientific result (position), the conclusion of the applicant, formulated in the dissertation**

The scientific results obtained in the dissertation work are new and obtained for the first time. On the topic of the thesis were published in 3 publications in scientific editions recommended by Committee for Control in Education and Science of RK for publication of the results of scientific activities, as well as 6 works in ranked scientific journals with impact factor indexed in the database Scopus and Thomson Reuters.

## 5 Theoretical and practical significance of the results

The results of the dissertation are mainly theoretical and fundamental, their scientific significance is due to the use of deep, modern mathematical results.

## 6 Comments and suggestions on the dissertation

For the author would like to wish to slightly change the way the material is presented. Before formulate and prove the results, the author should think about the motivating reasons for possible generalizations and the prospects for their application. This comment is of a recommendation nature and does not reduce the positive impression of the work as a whole.

## 7 The relevant content of the dissertation is within the framework of the requirements of the Rules for awarding academic degrees

A. Kassymov's dissertation was performed at a high scientific level and meets the requirements of the PhD. Based on the above, I believe that the work of A. Kassymov "Basic functional and geometric inequalities for the fractional order operators on homogenous Lie groups", in general, has the character of a completed study and meets the requirements for doctoral dissertations in the specialty "6D060100 - Mathematics", and its author deserves the award of the desired degree of doctor of philosophy (PhD).

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